Abstract
of the dissertation submitted by Mergul Oralbekovna Mustafina
for the degree of Doctor of Philosophy (PhD)
in the educational program 8D05401 – Mathematics,
entitled "Identification of Boundary Conditions for Differential Equations on Graph Trees"

General description of the study: The dissertation investigates the problems of unique reconstruction of boundary coefficients of Sturm–Liouville operators on graphs, as well as the coefficients included in the matching (compatibility) conditions, based on certain finite sets of eigenvalues of related operators.

Relevance of the study: Differential operators on metric graphs constitute a branch of mathematical physics that finds applications in nanophysics, wave propagation, chemistry, and other natural sciences. In this context, the models arising in [1] make it possible to analytically compute the spectral and scattering characteristics of operators while preserving the underlying geometric structure. Therefore, such models serve as an important tool for the theoretical investigation of spectral phenomena. Furthermore, in [2], differential operators on graphs are employed to analyze mechanical models consisting of strings and rods, electrical circuits, as well as hydraulic and thermal networks.

In this dissertation, partial inverse spectral problems for quantum graphs are considered. The main objective of inverse spectral analysis is to reconstruct an operator from its spectral characteristics. Such problems frequently arise in various applications when it is necessary to determine the properties of a medium described by the coefficients of a differential operator using observable data, such as spectral information. The fundamental results of the theory of inverse problems for differential operators defined on intervals can be found in the monographs [3–6]. A sufficiently comprehensive survey of inverse spectral problems for differential operators on graphs is presented in [7]. In addition, the study by V.A. Yurko [8] develops a constructive algorithm for the reconstruction of Sturm–Liouville potentials on arbitrary topological graphs.

State of the art of the study: We now turn to a review of results concerning partial inverse spectral problems for classical differential operators. Such problems aim at reconstructing the coefficients of a differential expression from spectral characteristics on a certain part of the domain (a finite interval or a geometric graph), while the coefficients on the remaining part are assumed to be known a priori. Typically, partial inverse problems require less spectral data than full inverse problems. They are also referred to as half-inverse problems, Hochstadt–Lieberman-type problems, inverse problems with mixed data, or incomplete inverse problems. In 1978, Hochstadt and Lieberman demonstrated that prior knowledge of the potential on a subinterval reduces the amount of spectral data required to reconstruct the operator. The Hochstadt–Lieberman problem is regarded as the first example of a partial inverse problem [9].

Subsequently, various generalizations of this problem were studied in the works of O. Hald [10], F. Gesztesy and B. Simon [11], M. Horváth [12, 13], and others. Conditions for solvability and constructive solution methods for the Hochstadt–Lieberman problem were obtained in the works of L. Sakhnovich [14], R.O. Hryniv and Yu.V. Mykytyuk [15], S.A. Buterin [16, 17], as well as O. Martynyuk and V.N. Pivovarchik [18, 19].

A wide range of partial inverse problems also arises for differential operators on geometric graphs, i.e., quantum graphs. Such operators are used to model various processes on graph-like structures and networks in organic chemistry, mesoscopic physics, nanotechnology, hydrodynamics, waveguide theory, and other applications [2, 20]. An introduction to quantum graphs can be found in the work of P. Kuchment [21]. There also exists an extensive body of literature devoted to inverse spectral problems for differential operators on graphs [7]. Here, we focus on partial inverse problems, which arise when the coefficients of differential operators (for example, Sturm–Liouville potentials) are known a priori on a certain part of the graph. These coefficients may be obtained either through measurements or by reconstruction methods. In the latter case, solutions of partial inverse problems serve as an intermediate step in solving full inverse problems on graphs.

The first results on partial inverse problems for graphs were obtained in the works of V.N. Pivovarchik [22], V.A. Yurko [23], and S. Yang, among others [24–26]. However, these results were limited to uniqueness theorems for Sturm–Liouville (Schrödinger) operators on graphs of simple structure (such as star graphs and simple graphs with loops). Later, N.P. Bondarenko developed a constructive method for solving partial inverse problems on graphs of various types. Using this approach, new results were obtained for star graphs [27–30], simple graphs with cycles [31, 32], tree graphs (acyclic graphs) [33], and even for differential operators and vertex conditions on graphs with arbitrary geometric structure [34, 35]. These results include not only uniqueness theorems but also constructive algorithms ensuring the solvability and stability of partial inverse problems.

Thus, the present dissertation is devoted to an in-depth study of methods for identifying boundary coefficients of Sturm–Liouville operators on graphs, as well as the coefficients appearing in matching conditions, based on certain finite sets of eigenvalues of related operators. The proposed identification methods possess significant theoretical and practical importance for various fields of science and engineering. At the initial stage, these methods are adapted to differential operators on tree-type and star-type graphs.

Aim and objectives of the study

The main aim of the dissertation is to investigate the problems of unique reconstruction of the domains of definition of Sturm–Liouville operators on tree graphs.

To achieve this aim, the following objectives are set:

– to study direct spectral problems for second-order differential operators on star graphs and tree graphs based on the analysis of their fundamental solutions and characteristic determinants;

– to derive asymptotic formulas for the eigenvalues of Sturm–Liouville operators on star graphs;

– to investigate identification problems for domains of definition of second-order differential operators on star and tree graphs as partial inverse spectral problems;

– to justify the possibility of unique reconstruction of boundary coefficients and coefficients appearing in matching conditions from finite sets of eigenvalues of related operators.

Object of the study: Linear differential operators on geometric graphs.

Subject of the study: Unique reconstruction of boundary coefficients and coefficients included in matching conditions from finite sets of eigenvalues of related operators.

Research methodology: The dissertation employs well-established methods of the general theory of differential equations, functional analysis, and the theory of functions of a complex variable. In particular, in the first chapter, the Horn method is used to determine approximate zeros of entire functions, while in the second chapter, residue theorems and properties of meromorphic functions are applied. In certain sections, Weinstein determinants are utilized, allowing for a more detailed analysis of the interlacing properties of zeros of entire functions.

As a result, it is proven that the sought coefficients can be uniquely reconstructed from a finite number of eigenvalues of related operators. At the same time, it is shown that such reconstruction is impossible when only a finite set of eigenvalues of a single initial operator is available.

Novelty of the study: The asymptotic formulas for eigenvalues obtained in the first chapter constitute a new result for the case where the lengths of graph edges are different. Previously, such asymptotics were considered only for zero potentials and graphs with edges of equal length. In this dissertation, the potentials are assumed to be differentiable functions, and the edge lengths are allowed to vary.

Another novel contribution of this work is the demonstration that only a finite number of eigenvalues of related operators is sufficient for the unique reconstruction of the unknown coefficients, as well as the proposal of a procedure for selecting such related operators.

Theoretical and practical significance of the study: The dissertation is of a theoretical nature in terms of the methods employed and the results obtained. The work extensively applies the Lagrange formula from the theory of differential equations, analytical methods from the theory of entire and meromorphic functions, as well as results from the spectral theory of linear operators. Despite its theoretical character, the results are directly relevant to applied problems. In particular, in the analysis of membrane vibrations, it is common to approximate membranes by one-dimensional networks, which can be modeled as graphs. Thus, two-dimensional problems can be approximated by one-dimensional graph models, reducing the dimensionality of the problem.

Therefore, the theoretical results of this work can serve as a methodological foundation for solving practical applied problems.

Main scientific contributions presented for defense:

– Direct spectral problems for second-order differential operators on star graphs and tree graphs were investigated based on a comprehensive analysis of their fundamental solutions and characteristic determinants;

– asymptotic formulas for the eigenvalues of Sturm–Liouville operators on star graphs were derived;

– identification problems for domains of definition were studied as partial inverse spectral problems within the framework of spectral analysis of second-order differential operators on star and tree graphs;

– the possibility of unique reconstruction of boundary coefficients and coefficients appearing in matching conditions from finite sets of eigenvalues of related operators was substantiated.

Relationship with other research projects:

The research is related to the implementation of scientific projects funded by the Science Committee of the Republic of Kazakhstan and the Ministry of Science and Higher Education. The author participated in the following project as a junior researcher:

- IRN No. AP19678089, “Inverse Spectral Problems for Differential Operators on Graphs” (2023–2025).

Author’s contribution. All scientific results presented in the dissertation were obtained with the direct participation of the author in collaboration with the scientific supervisors. The author actively contributed to the development of research methods, the solution of the stated problems, and the analysis of the obtained results.

The scientific supervisors contributed to the proper formulation of the problems, the identification of the main research directions, and the discussion of the results.

Approbation of the study and publications. The results of the dissertation were presented at international scientific conferences, including those held abroad, as well as at scientific seminars:

A total of 6 publications have been produced on the topic of the dissertation, including 3 papers published in international journals indexed in the Web of Science and Scopus databases, 1 paper published in a journal recommended by the Committee for Quality Assurance in Science and Higher Education of the Ministry of Science and Higher Education of the Republic of Kazakhstan, 2 papers published in the proceedings of international scientific conferences.

Publications:

1. Kanguzhin B.E., Kaiyrbek Zh.A., Mustafina M.O. Recovering of the Stiffness Coefficients of the Sturm–Liouville Operator on a Star Graph from a Finite Set of Its Eigenvalues// Lobachevskii Journal of Mathematics. -2024.-V.45, No. 6. – pp. 2711–2716. https://doi.org/10.1134/S1995080224602881 (WoS Q2, Scopus percentile 57). 02.10.2024.

2. Kanguzhin B.E., Kaiyrbek Zh.A., Mustafina M.O. Determination of Stiffness Coefficients at the Internal Vertices of the Tree Based on a Finite Set of Eigenvalues of an Asymmetric Second-Order Linear Differential Operator// Symmetry. -2025.-V.17, 1263 https://doi.org/10.3390/sym17081263 (WoS Q2, Scopus percentile 93). 07.08.2025.

3. Kanguzhin B.E., Mustafina M.O., Kaiyrbek O.A. Well-posed problems for the Laplace-Beltrami operator on a stratified set consisting of punctured circles and segments // Bulletin of the Karaganda University. Mathematics Series. -2025.-

V.119, №3. -P 150-163 https://doi.org/10.31489/2025M3/150-163 (WoS Q2, Scopus percentile 53). 30.09.2025.

4. G-H. Aimal Rasa, M. Madiyarov, M. Mustafina. On the unique solvability of nonlocal in time problemscontaining the ionkin operator in the spatial variable // KazNU Bulletin. Journal of Mathematics, Mechanics and Computer Science. - 2025.- V. 128, No.4 35-57. https://doi.org/10.26577/JMMCS202512844. 30.12.2025.

International Scientific Conference **"Ufa Autumn Mathematical School – 2023"** (Ufa University of Science and Technology, Ufa, Russian Federation, 2023);

International Scientific and Practical Conference "Transformation of Education and Science as a Key Factor in Improving Human Capital Quality" (Amanzholov Readings – 2023), dedicated to the 120th anniversary of Sarsen Amanzholov (S. Amanzholov East Kazakhstan University, Oskemen, 2023);

Republican Scientific and Practical Conference "The Scientific Heritage of Zaki Akhmetov and National Values" dedicated to the 95th anniversary of Zaki Akhmetov (S. Amanzholov East Kazakhstan University, Oskemen, 2023);

International Scientific Conference **"Ufa Autumn Mathematical School – 2024"** (Ufa University of Science and Technology, Ufa, Russian Federation, 2024);

International Scientific Conference **"Inverse Problems in Natural Sciences"** (S. Amanzholov East Kazakhstan University, Oskemen, 2024);

X International Scientific and Practical Conference of Students, Master's and PhD Students, and Young Scientists "Integration of Education, Science, and Business: Current Issues and Prospects" dedicated to the 125th anniversary of Kanysh Satbayev (S. Amanzholov East Kazakhstan University, Oskemen, 2024);

International Scientific and Practical Conference "Science, Society and Artificial Intelligence: Current Issues of Interaction" (Amanzholov Readings – 2025) (S. Amanzholov East Kazakhstan University, Oskemen, 2025);

International Scientific Conference **"Ufa Autumn Mathematical School – 2025"** (Ufa University of Science and Technology, Ufa, Russian Federation, 2025).

Structure and volume of the dissertation. The dissertation consists of an introduction, four chapters, appendices, a conclusion, and a list of references. The total volume of the work is 109 pages. The bibliography includes 136 references.

The introduction substantiates the relevance of the research topic and outlines the aims, objectives, scientific novelty, and structure of the dissertation.

The conclusion summarizes the main results of the study, highlights their scientific and practical significance, and outlines directions for future research.

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